Putnam Problems — Prof. Madras

November 4, 2024

The Cauchy-Schwarz Inequality says that for every integer $n \geq 1$ and all real numbers $a_1, \dots, a_n$ and $b_1, \dots, b_n$,

$$a_1b_1 \,+\, a_2b_2 \,+\, \cdots \,+\, a_nb_n \;\leq\; \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}\,\sqrt{b_1^2 + b_2^2 + \cdots + b_n^2}\,,$$

or equivalently

$$\left(a_1b_1 \,+\, a_2b_2 \,+\, \cdots \,+\, a_nb_n\right)^2 \;\leq\; \left(a_1^2 + a_2^2 + \cdots + a_n^2\right)\,\left(b_1^2 + b_2^2 + \cdots + b_n^2\right)\,,$$

and that equality holds if and only if there is a real constant λ such that $a_1 = \lambda b_1$, $a_2 = \lambda b_2$, $\dots a_n = \lambda b_n$ (or $b_1 = \lambda a_1$, $\dots b_n = \lambda a_n$).

In terms of vectors in $\mathbb{R}^n$: if we write $\vec{v} = (a_1, a_2, \dots, a_n)$ and $\vec{w} = (b_1, b_2, \dots, b_n)$, then the "dot product" of these two vectors is $\vec{v} \cdot \vec{w} = a_1b_1 + \cdots + a_nb_n$ and their Euclidean norms (or lengths) are $\|\vec{v}\| = \sqrt{a_1^2 + \cdots + a_n^2}$ and $\|\vec{w}\| = \sqrt{b_1^2 + \cdots + b_n^2}$. Thus the Cauchy-Schwarz Inequality says

$$\vec{v} \cdot \vec{w} \;\leq\; \|\vec{v}\|\,\|\vec{w}\|$$

with equality if and only if $\vec{v} = \lambda\vec{w}$ for some real number λ (or $\vec{w} = \lambda\vec{v}$).

Here are some problems that use the Cauchy-Schwarz Inequality.

CS.1. Prove that for all positive real numbers a, b, and c,

$$9a^2b^2c^2 \;\leq\; (a^2b + b^2c + c^2a)\,(ab^2 + bc^2 + ca^2)\,.$$

CS.2. Prove that for all positive real numbers $a_1, \dots, a_n$,

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \;\geq\; \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}}\,.$$

Remark: The left-hand expression is called the "arithmetic mean" and the right-hand expression is called the "harmonic mean." It also true that

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \;\geq\; (a_1a_2\cdots a_n)^{1/n} \;\geq\; \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}}\,,$$

with $(a_1a_2\cdots a_n)^{1/n}$ called the "geometric mean." The inequality between the arithmetic and geometric means is also very useful.

CS.3. Prove that for all positive x, y, and z,

$$\sqrt{\frac{x+y}{x+y+z}} + \sqrt{\frac{x+z}{x+y+z}} + \sqrt{\frac{y+z}{x+y+z}} \;\le\; \sqrt{6}\,.$$

CS.4. Prove that for all positive x, y, and z,

$$x+y+z \;\le\; 2\left(\frac{x^2}{y+z} + \frac{y^2}{x+z} + \frac{z^2}{x+y}\right).$$

CS.5. Assume $a_1+a_2+\cdots+a_n \;=\; n$. Prove that $a_1^4+a_2^4+\cdots+a_n^4 \;\ge\; n$. (Hint: First consider $a_1^2+a_2^2+\cdots+a_n^2$.)

CS.6. Let $P(x)$ be a polynomial with nonnegative coefficients. Prove that

$$\sqrt{P(a)P(b)} \;\ge\; P(\sqrt{ab})$$

for all positive a and b.

CS.7. Find the minimum value of $x^2+y^2+z^2$ subject to the constraint $x+2y+3z=7$. (Instead of using multivariable calculus, use the Cauchy-Schwarz Inequality together with its "if and only if" part.)