Putnam Problems — Prof. Madras

November 18, 2024

Continuing from last week, this week's problems involve polynomials. Here are some basic facts about polynomials (revised).

1. Let $P(x) = a_k x^k + a_{k-1}x^{k-1} + \cdots + a_1 x + a_0$ with $a_k \neq 0$. Then P has k roots $r_1, \ldots . r_k$ (some of which may be repeated), and we have

$$P(x) = a_k(x - r_1)(x - r_2)\cdots(x - r_k).$$

Moreover, expanding this expression shows that

$$\begin{aligned} r_1 + r_2 + \cdots + r_k &= -\frac{a_{k-1}}{a_k}, \\ r_1r_2 + r_1r_3 + \cdots + r_{k-1}r_k &= \frac{a_{k-2}}{a_k} \\ &\vdots \\ r_1r_2\cdots r_k &= (-1)^k\frac{a_0}{a_k}. \qquad \text{“Viète's relations”} \end{aligned}$$

2. Assume two polynomials $f(x)$ and $g(x)$ of degree at most k satisfy $f(x) = g(x)$ for at least $k+1$ values of x. Then $f(x) = g(x)$ for every x.
(*Proof:* The assumption implies that $f(x) - g(x)$ is a polynomial of degree at most k with at least $k+1$ roots, which can only happen if it is identically 0.)
Corollary: Assume two polynomials $f(x)$ and $g(x)$ satisfy $f(x) = g(x)$ for infinitely many values of x. Then $f(x) = g(x)$ for every x.

P.2: (From last week) Let $P(x)$ be a polynomial of degree 99, and assume

$$P(1) = 1, \quad P(2) = \frac{1}{2}, \quad P(3) = \frac{1}{3}, \quad \ldots, \quad P(100) = \frac{1}{100}.$$

Evaluate $P(101)$.
(*Hint*: What do you know about the polynomial $xP(x) - 1$?)

P.3. Let r_1 and r_2 be the roots of the equation $x^2 + 2x + 3 = 0$. Without explicitly solving for r_1 and r_2, evaluate

$$\frac{1}{r_1} + \frac{1}{r_2}.$$

P.4. Let a, b, and c be real numbers. Prove that these three numbers are all nonnegative if and only if $a + b + c \geq 0$, $ab + bc + ac \geq 0$, and $abc \geq 0$.

P.5. Find all polynomials $p(x)$ such that $xp''(x) + p(x) = x^2 + 1$ (for all x).