Putnam Problems — Prof. Madras

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To start this week, here is a recent Putnam problem to try.

2021-A1. A grasshopper starts at the origin in the coordinate plane and makes a sequence of hops. Each hop has length 5, and after each hop the grasshopper is at a point whose coordinates are both integers; thus, there are 12 possible locations for the grasshopper after the first hop. What is the smallest number of hops needed for the grasshopper to reach the point (2021, 2021)?

(*Comment:* There are two things that you should do here. First, figure out what the answer should be. Then prove that your answer is correct.)

The remaining problems this week involve polynomials. Here are some important basic facts about polynomials.

1. A polynomial (which is not identically zero) of degree k has exactly k complex roots (some may be repeated roots). If $P(x)$ is a polynomial of degree k and its roots are $r_1, \ldots . r_k$, then

$$P(x) \;=\; A(x-r_1)(x-r_2)\cdots(x-r_k),$$

where A is the coefficient of x^k in $P(x)$.

2. Assume two polynomials $f(x)$ and $g(x)$ satisfy $f(x) = g(x)$ for infinitely many values of x. Then $f(x) = g(x)$ for every x.
(*Proof:* The assumption implies that $f(x) - g(x)$ is a polynomial with infinitely many roots, which can only happen if it is identically 0.)

P.1: (a) Determine all polynomials P such that $P(0) = 0$ and $P(x+1) = P(x) + 1$ for every x.
(b) Determine all polynomials P such that $P(0) = 0$ and $P(x^2+1) = [P(x)]^2 + 1$ for every x.

P.2: Let $P(x)$ be a polynomial of degree 99, and assume

$$P(1) = 1, \quad P(2) = \frac{1}{2}, \quad P(3) = \frac{1}{3}, \quad \ldots, \quad P(100) = \frac{1}{100}.$$

Evaluate $P(101)$.