Putnam Notes — Prof. Madras

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On Monday (Sept 30), we were looking at the following problem:

4. Assume that the function $f : \mathbb{N} \to \mathbb{N}$ satisfies

$$x\,f(y) \,+\, y\,f(x) \;=\; (x+y)\,f(x^2+y^2) \qquad \text{for all } x,y \in \mathbb{N}.$$

Prove that f must be a constant function.

The solution we ended up with was somewhat roundabout. A more direct solution is based on the following observation:

Observation: If $f(x) < f(y)$, then $f(x) < f(x^2+y^2) < f(y)$.

To prove this, suppose $f(x) < f(y)$. Then

$$f(x^2+y^2) \;=\; \frac{x\,f(y) \,+\, y\,f(x)}{x+y} \;<\; \frac{x\,f(y) \,+\, y\,f(y)}{x+y} \;=\; f(y)$$

and similarly

$$f(x^2+y^2) \;=\; \frac{x\,f(y) \,+\, y\,f(x)}{x+y} \;>\; \frac{x\,f(x) \,+\, y\,f(x)}{x+y} \;=\; f(x).$$

Now think about what this means. It says that between every two numbers in the range of f, there is another number strictly between them that is also in the range. Since the range is a subset of $\mathbb{N}$, this should lead to a contradiction.

Specifically, assume that f is not constant. Then the range of f contains at least two distinct numbers. Let M_1 be the smallest number in the range and let M_2 be the second-smallest number in the range of f. Let x_1 and x_2 be positive integers such that $f(x_1) = M_1$ and $f(x_2) = M_2$. Then the observation tells us that $M_1 < f(x_1^2 + x_2^2) < M_2$, which contradicts that fact that M_2 is the second-smallest number in the range.

The contradiction proves that f must only have one value in its range, i.e. that f is constant. Q.E.D.

On September 23, we mentioned the *Pigeonhole Principle:*

If n items are put into m boxes, and if $n > m$, then at least one box must contain more than one item.

Here are a couple of problems that can be solved using the Pigeonhole Principle.

PP1. (a) Show that in any set of three integers $\{x_1, x_2, x_3\}$, there must be two of them whose average is an integer.
(b) Show that in any set of five ordered pairs of integers $\{(x_i, y_i), i = 1, 2, 3, 4, 5\}$. there must be two of them whose average is another ordered pair of integers.

PP2. You have a 3×7 grid of squares. Each square is coloured red or blue. Show that there is a sub-rectangle of the grid whose four corner squares all have the same colour.
(*Suggestion:* Think about the number of possible coloured columns. Why would this be easier if we changed 3×7 to 3×9?)