Putnam Practice Problems — Prof. Madras – Fall 2024

Here are some warm-up problems to try. They are somewhat easier than most Putnam problems. We shall look at them during our first session or two.

I use the standard notation that $\mathbb{N}$ is the set of natural numbers $\{1, 2, 3, \ldots\}$.

1. Prove that the product of four consecutive integers (e.g. 8, 9, 10, 11) must be one less than a perfect square.

2. Let S be a set of 50 positive integers, each of which is less than 99. Show that there exist x and y in S such that $x + y = 99$.

3. Let $x_1, x_2, x_3, \ldots$ be a sequence of positive real numbers that satisfy

$$x_1 = 1 \quad \text{and} \quad x_{n+1}^2 + x_{n+1} \;=\; x_n \quad \text{for every } n \in \mathbb{N}.$$

Prove that $x_n \geq \frac{1}{n}$ for every $n \in \mathbb{N}$.

4. Assume that the function $f : \mathbb{N} \to \mathbb{N}$ satisfies

$$x\, f(y) \;+\; y\, f(x) \;=\; (x+y)\, f(x^2+y^2) \quad \text{for all } x, y \in \mathbb{N}.$$

Prove that f must be a constant function.

5. Let p and q be positive integers. On an arbitrarily large chessboard, one move of a (p, q)-knight jumps $\pm p$ squares in the horizontal or vertical direction and $\pm q$ steps in the perpendicular direction. (Thus a $(1, 2)$-knight is an ordinary chess knight.) Prove that a (p, q)-knight can return to its original position only after an *even* number of steps.

6. There is a pile of 1001 stones on a table. You can perform the following operation repeatedly: choose one pile containing more than one stone, throw away one stone from the pile, and then divide that pile into two smaller (not necessarily equal) piles. Is it possible to reach a situation in which each pile on the table has exactly three stones?