On Totally-Concave Polyominoes*

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Abstract

A polyomino is an edge-connected set of cells on the square lattice. Every row or column of a *totally-concave* (TC) polyomino consists of more than one sequence of consecutive cells of the polyomino. We show that the minimum area (number of cells) of a TC polyomino is 21 cells. We also suggest, implement, and run an efficient algorithm for counting TC polyominoes. Finally, we prove that the associated sequence $(\kappa(n))$ has a finite growth constant λ_κ, prove the lower bound $\lambda_\kappa > 2.4474$, and conjecture that λ_κ is equal to the growth constant of *all* polyominoes.

1 Introduction

A *polyomino* of area n is a connected set of n cells on the square lattice $\mathbb{Z}^2$, where connectivity is through edges. Two polyominoes are considered equivalent if one can be transformed into the other by a translation.

Counting polyominoes is a long-standing problem in discrete geometry, originating in statistical physics in the context of percolation processes [10] and popularized in Golomb's pioneering book [12] and by M. Gardner's columns in *Scientific American*. The sequence $A(n)$, which lists the number of polyominoes, is currently known up to $n = 70$ [1].

The growth constant of polyominoes has also attracted much attention in the literature. Klarner [16] showed that the limit (a.k.a. *Klarner's constant*) $\lambda := \lim_{n\to\infty} \sqrt[n]{A(n)}$ exists. The convergence of $A(n+1)/A(n)$ to λ, as $n \to \infty$, was proved only three decades later by Madras [17]. The best-known lower [4] and upper [5] bounds on λ are 4.0025 and 4.5252, respectively. By applying numerical methods to the known values of $A(n)$, it is widely believed that $\lambda \approx 4.06$, and the currently best estimate of λ is 4.0625696 ± 0.0000005 [15]. (Based on the new counts of $A(n)$ till $n = 70$, a better estimate, 4.06256912(2), was provided to us by I. Jensen in a personal communication.)

In a *convex* polyomino, each row and column consists of exactly one maximal continuous sequence of cells. These polyominoes are essential in many application domains, and they attracted a considerable amount of attention in the literature. See, for example, a discussion of the asymptotic number of convex polyominoes [8], a derivation of a rather complex generating function for the sequence that enumerates convex polyominoes [9], a method for generating random convex polyominoes [13], and an investigation of the relation between ordering and convex polyominoes [11], among many other works.

However, the complement type of polyominoes was hardly considered. In a *totally-concave* (TC) polyomino, each row and column consists of at least *two* maximal continuous sequences of cells, as is shown in Figure 1.[1] It is hinted in Ref. [7, §14, p. 369, problem 14.5.4] that the minimum possible area of a TC polyomino is 21. Let $\kappa(n)$ be the number of TC polyominoes of size (area) n. An algorithm for computing $\kappa(n)$, for a given n, is also sought as an open problem [7, §14, p. 369, problem 14.5.5]. Among other results, we settle the minimality conjecture and suggest an efficient algorithm.

The paper is organized as follows. In Section 2, we prove that there do not exist TC polyominoes of area less than 21. In Section 3, we present a nontrivial extension of Jensen's algorithm to counting TC polyominoes, and report counts of these polyominoes up to size 35. In Section 4, we prove that the sequence $\kappa(n)$ has a growth constant λ_κ, prove that $\lambda_\kappa > 2.4474$, and provide a motivation for the conjecture that $\lambda_\kappa = \lambda$. We end in Section 5 with some concluding remarks and future research directions.

2 Minimum Area

Theorem 1 *The minimum area of a TC polyomino is 21.*

The proof of this theorem follows a necessity-sufficiency format. Necessity is shown by deducing upper and lower bounds on the area of TC polyominoes in $m \times \ell$ bounding boxes; These bounds contradict each other for areas less than 21. Sufficiency is evident by example.

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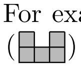
[1] Recipe for the picture in Figure 1(a.2) is available upon request.

(a.1) Symbolic (a.2) Eatable With marked edges
(a) Area 21 (b) Area 24

Figure 1: TC polyominoes of various areas and flavors. The symbolic representation in (b) distinguishes between *hidden* edges (green), *inside* edges (blue), and *outside* edges (red).

Proof. A lower bound on the area of a TC polyomino within an $m \times \ell$ bounding box is achieved by partitioning the edges of such a polyomino into *hidden*, *outside*, and *inside* edges, as shown in Figure 1(b). The top (resp., right/bottom/left) edge of a cell c is *hidden* if there is a cell of the polyomino immediately above (resp., to the right of/below/to the left of) c. An edge is *outside* if it is not facing any other edge. An *inside* edge is an edge facing another edge, but not immediately, that is, with a gap of at least one cell. Consider a TC polyomino. Denote by n its area, and by i, o, and h the number of inside, outside, and hidden edges, respectively, of the polyomino. For example, by these definitions, the "U-pentomino" () has $i = 2$, $o = 10$, and $h = 8$. For the area-24 TC-polyomino depicted in Figure 1(b), we have $i = 26$, $o = 24$, and $h = 46$. By pairing inside and outside edges in rows and columns, we have that $o = 2m + 2\ell$ and $i \geq 2m + 2\ell$. We also have that $h \geq 2n - 2$ since the polyomino is connected and, hence, it must include at least $n-1$ cell adjacencies. Since $h + o + i = 4n$, we have that $n \geq 2m + 2\ell - 1$.

For an upper bound on n, we may assume without loss of generality that $m \leq \ell$. Then, a TC polyomino within an $m \times \ell$ bounding box must be missing at least one cell from each of the ℓ columns, none of which is in the top or bottom row (for guaranteeing concavity of the columns), as well as at least two further cells, one in the top and one in the bottom row (for guaranteeing concavity of these rows). Therefore, $n \leq m\ell - \ell - 2$.

Altogether, we have that $2m+2\ell-1 \leq n \leq m\ell-\ell-2$, with $m \leq \ell$. A simple case analysis shows that the smallest n satisfying these constraints is 21, with $m = 5$ and $\ell = 6$.

Hence, $n \geq 21$ is a necessary condition for a TC polyomino. On the other hand, the existence of a TC polyomino of area 21 is evident by Fig. 1(a). This completes the proof. □

This result was confirmed by our TC-polyomino counting programs (see Section 3). Figure 2 shows representatives of the 152 TC polyominoes of area 21.

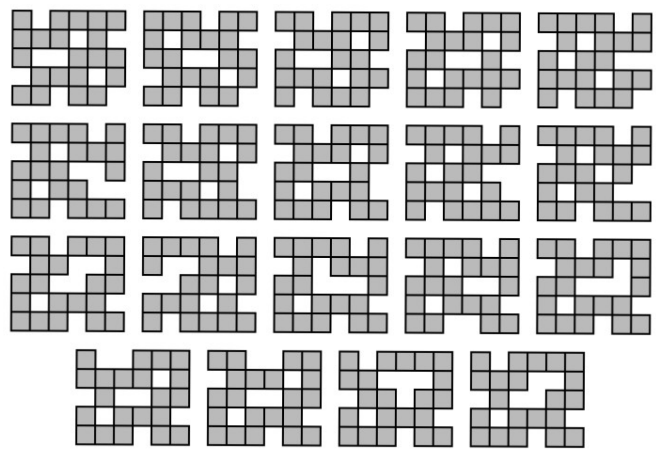


Figure 2: The 19 TC polyominoes of area 21, up to rotation and mirroring.

(None of these polyominoes have any symmetries, hence, the polyominoes formed by the eight orientations of each of the 19 drawn polyominoes are distinct.)

3 An Efficient Counting Algorithm

3.1 Algorithm

We first implemented a prototype backtracking algorithm for counting TC polyominoes. The program recursively concatenated concave columns to a growing polyomino. A branch of this procedure was abandoned when the area of the polyomino grew too large or if it was no longer possible for it to become connected with the addition of further columns. (This happened when a component of the polyomino became permanently detached.)

We then designed a much more efficient algorithm, based on Jensen's algorithm for counting all polyominoes [14, 15]. In a nutshell, Jensen's algorithm counts polyominoes within horizontal bounding strips of height h, where $1 \leq h \leq \lceil n/2 \rceil$. The algorithm considers column by column from left to right, and cell by cell from top to bottom within each column. At each cell, the algorithm considers either to have it occupied (belonging to the polyomino) or empty (not belonging). At all stages, the algorithm does not keep in memory all polyominoes but all possible *right boundaries* of polyominoes, that is, all combinations of the last h cells considered. The algorithm maintains a database whose entries have keys that are the different *signatures*, where a signature consists of a boundary plus all possible connections between cells on the boundary by cells found to the left of it. In other words, the keys reflect all possible splits of boundary cells into connected components, where the connections are to the left of the boundary. In addition, a signature also includes two bits that indicate whether or not the polyominoes associated with that entry touch the top and/or bottom of the strip. The contents of each entry in the database is statistics of all partially-built polyominoes ("partially" means that polyominoes may still consist of more than one connected component), that is, the *counts* of all polyominoes parameterized by area, having that specific signature. When the currently considered cell is chosen to be occupied, the counts of polyominoes are updated by adding the numbers of fully-built polyominoes, that is, polyominoes that consist of exactly one connected component and touch the top and bottom of the strip.

Our modifications. For counting TC polyominoes, we also need to ensure that each column and each row consists of more than one consecutive sequence of cells. This is simple to achieve for columns: At the end of processing a column, we discard from the database all entries that correspond to columns that contain less than two sequences of occupied cells. For rows, we enhance the signatures by splitting each one into at most 4^h subsignatures: For each row, we keep a code as follows: '0' indicates that the first sequence of occupied cells has not been met yet; '1' means that we are in the middle of the first sequence; '2' states that we are between the first and second sequences; and '3' signifies that we have already entered the second sequence. (Once we reach '3,' we do not need to update this indicator any more.) Figure 3 shows an enhanced signature, in which each boundary cell is associated with two numbers: The original vertical code (left), and the additional horizontal code (right). Then, we count only polyominoes with signatures whose line indicators are all '3.' Note that the indicators of the top and bottom rows make the two bits described above redundant.

Jensen's algorithm is efficient in the sense that it is the only known algorithm whose running time, $\tilde{O}(1.732^n)$ [3], is smaller than the total number of polyominoes, $\tilde{\Theta}(\lambda^n)$. (Recall that $\lambda \approx 4.063$.) Our modification splits every signature into at most $4^{n/2} = 2^n$ subsignatures (in practice, into much less than that), thus, the running time of the modified algorithm is $\tilde{O}(3.464^n)$, which is still much smaller than the total number of polyominoees. In conclusion, our version of the algorithm is slower than the original algorithm, although we eventually count fewer polyominoes, due to the exponential growth in the number of processed signatures.

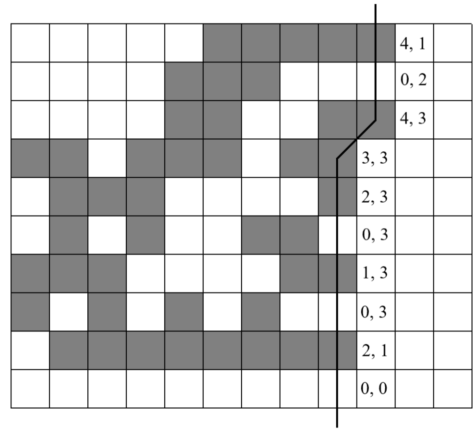


Figure 3: An enhanced boundary signature in the modified version of Jensen's algorithm.

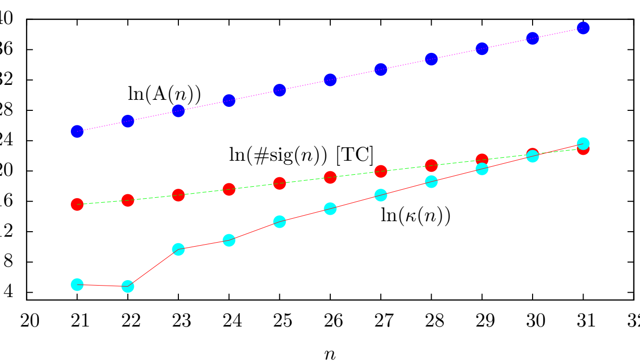


Figure 4: Plots of the number of signatures (while counting TC polyominoes), all poyominoes, and TC polyominoes.

Figure 4 plots in a semi-logarithmic scale the number of distinct signatures encountered by the algorithm while computing $\kappa(n)$) (in red circles), together with the number of TC polyominoes (cyan) and the total number of polyominoes (blue), all as functions of n, for $21 \leq n \leq 31$.

3.2 Results

Our prototype program, implemented in Python, computed in 90 hours (elapsed time) $\kappa(n)$ up to $n = 26$ on a PC with a 64-bit system operating an i5-9400F Intel Core CPU at 2.90GHz with 12GB of RAM.

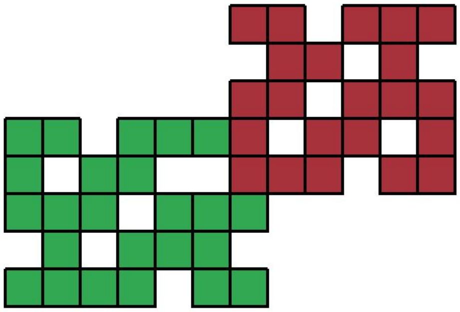


Figure 5: The concatenation of two TC polyominoes is always a TC polyomino.

The modified version of Jensen's algorithm was implemented in `C++` and run on a 12th generation Intel(R) i9-12900KF with 128GB of RAM. Using about 41 hours of CPU, the program computed $\kappa(n)$ up to $n = 35$, obtaining the values reported in Table 1 and agreeing with all values computed by the prototype program.

4 Growth Constant

Bender [8] showed that the number of convex polyominoes of size n is asymptotically $t\gamma^n$, for $\gamma \approx 2.3091$ and $t \approx 2.6756$, that is, the growth constant (see a formal definition below) of convex polyominoes is roughly 2.3091. In this section, we investigate the growth constant of TC polyominoes.

4.1 Existence

Definition 1 *(lexicographic order) For cells c_1, c_2, we say that $c_1 \prec c_2$ if c_1 lies in a column which is to the left of the column of c_2,* or *if c_1 lies below c_2 in the same column.*

Definition 2 *(concatenation) Let P_1, P_2 be two polyominoes, and let c_1 (resp., c_2) be the largest (resp., smallest) cell of P_1 (resp., P_2). The* concatenation *of P_1 and P_2 is the placement of P_2 relative to P_1, such that c_2 is found immediately on top of c_1.*

Figure 5 shows the concatenation of two polyominoes P_1 and P_2. The result of concatenating P_1 and P_2 is always a valid polyomino since the two polyominoes touch each other but do not overlap. Moreover, if both P_1 and P_2 are TC, then the result of concatenating them is also TC.

Theorem 2 *The limit $\lambda_\kappa := \lim_{n\to\infty} \sqrt[n]{\kappa(n)}$ (the growth constant of $(\kappa(n))$) exists and is finite.*

Proof. We follow the proof of existence and finiteness of Klarner's constant λ [16]. First, the sequence $\kappa(n)$ is supermultiplicative, that is, $\kappa(n)\kappa(m) \le \kappa(n+m)$ for all $m, n \in \mathbb{N}$. This is justified by a simple concatenation argument. Indeed, all TC polyominoes of area n can be concatenated with all TC polyominoes of area m (see, *e.g.*, Figure 5), yielding distinct TC polyominoes of area $n + m$. Second, there exists a constant $\mu > 0$ for which $\kappa(n) \le \mu^n$ for all $n \in \mathbb{N}$. For example, $\mu = \lambda$, the growth constant of all polyominoes. (This follows immediately from the fact that $\kappa(n) \le A(n) \le \lambda^n$.) By a lemma of Fekete (Klarner cites Ref. [18, p. 852] for similar results), the claim follows. □

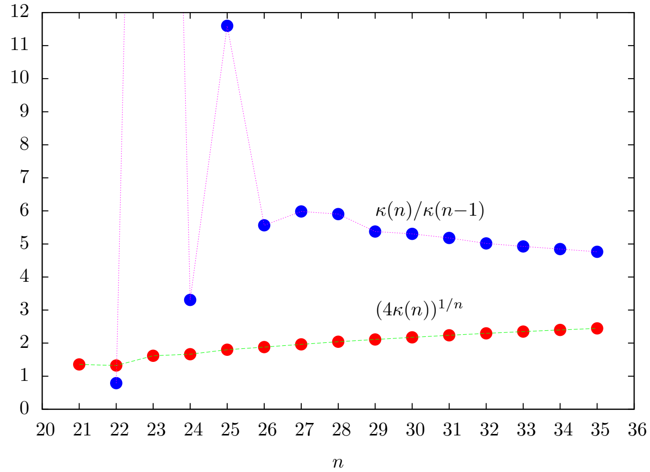


Figure 6: Plots of known values of $(4\kappa(n))^{1/n}$ and $\kappa(n)/\kappa(n-1)$.

It would be much more ambitious to prove the existence of the ratio sequence, that is, $\lim_{n\to\infty} \frac{\kappa(n)}{\kappa(n-1)}$. Obviously, if it exists, it must be equal to λ_κ.
Remark In fact, it makes more sense (see Section 4.2) to explore $((4\kappa(n))^{1/n})$ instead of $((\kappa(n))^{1/n})$. Figure 6 shows plots of the known values of $(4\kappa(n))^{1/n}$ and $\kappa(n)/\kappa(n-1)$. Surprisingly, the ratio sequence seems empirically to be monotone *decreasing* (except some low-order fluctuations), a property rarely found in other families of polyominoes.

4.2 Lower Bound

We now present a computer-assisted proof of a lower bound on λ_κ.

Definition 3 *(composition) A* composition *of two polyominoes is a relative placement of the two polyominoes, such that they touch (edge to edge), possibly in multiple places, but do not overlap.*

Figure 7 shows a few compositions of a pair of polyominoes P, Q. Some compositions (*e.g.*, those shown in Figures 7(b-d)) are *lexicographic*, that is, compositions in which all cells of P are lexicographically smaller than

Table 1: Counts of TC polyominoes.

n	$\kappa(n)$	n	$\kappa(n)$	n	$\kappa(n)$	n	$\kappa(n)$
1–20	0	24	52,306	28	119,309,768	32	88,476,873,440
21	152	25	606,636	29	641,447,812	33	435,921,253,072
22	120	26	3,376,528	30	3,403,173,276	34	2,113,011,155,472
23	15,820	27	20,204,672	31	17,634,751,456	35	10,065,872,407,536

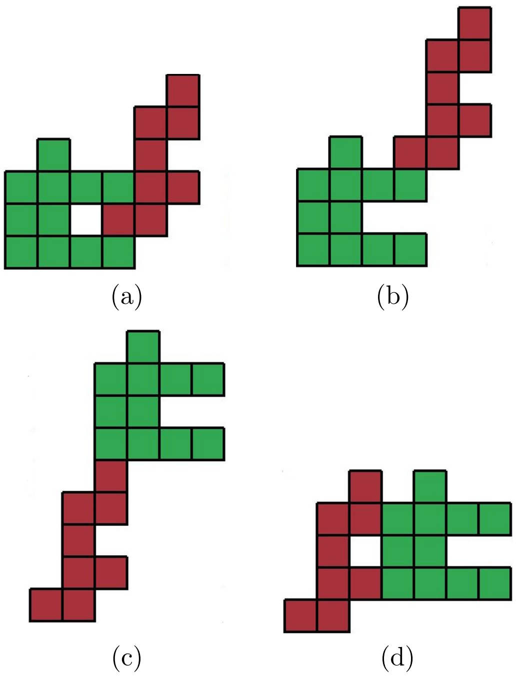


Figure 7: A few compositions of a sample pair of polyominoes.

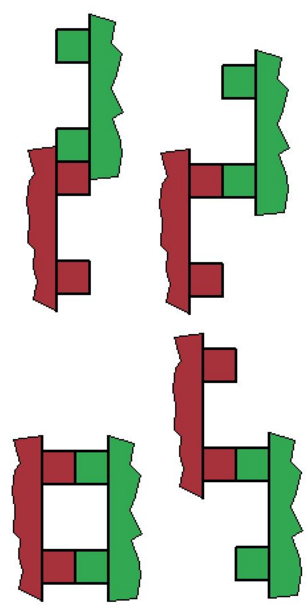


Figure 8: There are at least four lexicographic compositions of any pair of TC polyominoes.

all cells of Q (or vice versa), while other compositions (see, *e.g.*, Figure 7(a)) are not lexicographic. It is easy to observe that a composition of two TC polyominoes is not always a TC polyomino. However, any *lexicographic* composition of two TC polyominoes is also TC.

Lemma 3 *(A simplified version of Theorem 1(a) in Ref. [2, p. 3]) Assume that the limit* $\mu := \lim_{n\to\infty} \sqrt[n]{Z(n)}$ *exists for a sequence* $(Z(n))$*. Let* $c_1 > 0, c_2$ *be some constants. Then, if* $c_1 n^{c_2} Z^2(n) \leq Z(2n)\ \forall n \in \mathbb{N}$*, then* $\sqrt[n]{c_1(2n)^{c_2} Z(n)} \leq \mu\ \forall n \in \mathbb{N}$.

Theorem 4 $\lambda_\kappa > 2.4474$.

Proof. We use a composition argument, using the property that the extreme (rightmost and leftmost) columns of any TC polyomino have at least two cells. This property allows at least four lexicographic compositions of any pair of TC polyominoes P, Q that yield TC polyominoes. It can easily be verified that the minimum number of such compositions is obtained when both the rightmost column of P and the leftmost column of Q contain exactly two cells, with the same vertical gap between them. For such pairs of TC polyominoes, we have the four lexicographic compositions shown in Figure 8. Indeed, if the gaps between these cells are different (as seen in Figure 9), the two TC polyominoes P, Q have five lexicographic compositions; and if the respective columns of P, Q have more than two occupied cells, the number of lexicographic compositions may only increase.

Consequently, we have that $4(\kappa(n))^2 \leq \kappa(2n)$. Then, Lemma 3 implies that any term of the form $(4\kappa(n))^{1/n}$ is a lower bound on λ_κ. Checking the known values of $\kappa(n)$, we see that $n = 35$ provides the best lower bound $\lambda_\kappa \geq (4\kappa(35))^{1/35} > 2.4474$. □

4.3 Conjectured Value

Figure 4 may suggest that the growth constant of TC polyominoes is identical to that of all polyominoes. We state this as a conjecture and provide for it a tentative proof that depends on another well-known conjecture about the average diameter of lattice animals.

Conjecture 1 $\lambda_\kappa = \lambda$.

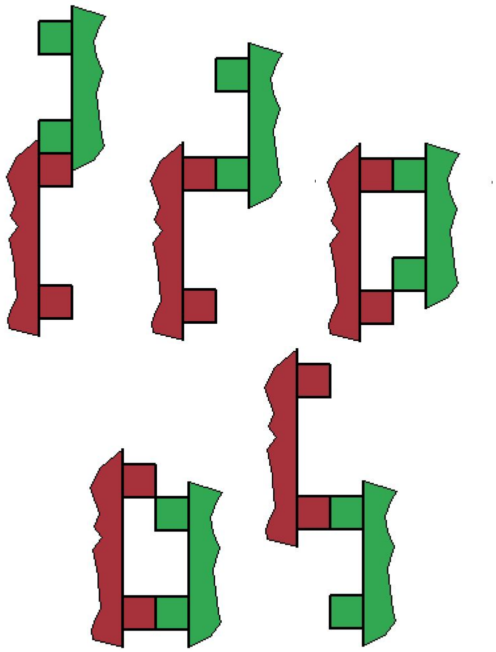


Figure 9: Five lexicographic compositions of a pair of TC polyominoes.

It is generally *believed* [19, §9.2] that all standard models of lattice animals and lattice trees (including polyominoes) have average diameter that scales as n^ν for some critical exponent $\nu < 1$ that depends only on the dimension of the lattice. Numerically, $\nu \approx 0.64$ in two dimensions. This is borne out in several numerical and theoretical studies in the physics literature. Here we can define the "diameter" of a polyomino P as the maximum Euclidean distance between any two cells of P. In particular, since (it is believed that, say) $\nu < 0.9$, let U_n be the set of all polyominoes of size n whose diameter is less than $n^{0.9}$. Then, the above belief implies that $|U_n| > A(n)/2$ for all sufficiently-large n.

Refer to Figure 10. Let L be the L-shaped frame depicted in red in the figure. Its width and height are $n^{0.9}$. Let $\alpha(n)$ be the number of cells in L. Then, $\alpha(n) = \Theta(n^{0.9})$. For any polyomino $P \in U_n$, let $f(P)$ be the union of L with the translation of P (colored in green) that has the lower left corner of its bounding box at (0,0). Then, $f(P)$ is a TC polyomino, and its area is $n+\alpha(n)$. Since the function $f(\cdot)$ is clearly one-to-one, we deduce that $\kappa(n + \alpha(n)) \geq |U_n|$. It follows that

$$\lambda_\kappa^{n+\alpha(n)} \geq \kappa(n + \alpha(n)) \geq A(n)/2$$

for all sufficiently-large n. Now take nth roots of the above, and let $n \to \infty$. The leftmost side converges to λ_κ, and the rightmost side converges to λ. We conclude that $\lambda_\kappa \geq \lambda$. The reverse relation is trivial, hence, $\lambda_\kappa = \lambda \approx 4.06$.

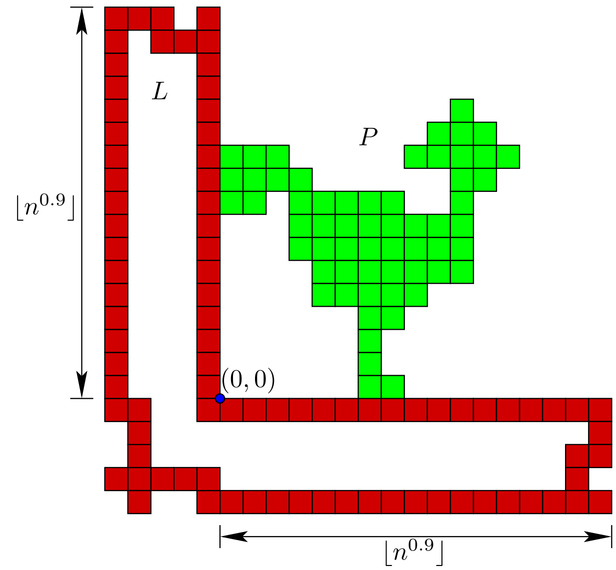


Figure 10: The function $f(P)$.

To the best of our knowledge, if this conjecture were true, then the family of TC polyominoes would be the only nontrivial proper subset of polyominoes previously studied in the literature that has been shown to have the same growth constant as all polyominoes.

5 Conclusion and Future Work

In this paper, we investigate a few problems related to TC polyominoes. We prove that the minimum possible area of such a polyomino is 21; suggest an efficient algorithm for counting TC polyominoes, and report counts of TC polyominoes till area 35; show that $(\kappa(n))$, the sequence of counts of TC polyominoes of area n, has a growth constant λ_κ; prove that $\lambda_\kappa > 2.4474$; and finally, conjecture that $\lambda_\kappa = \lambda \approx 4.06$.

Our main future research directions are the following.

1. Prove the existence of the limit of the ratio sequence, that is, $\lim_{n\to\infty} \frac{\kappa(n)}{\kappa(n-1)}$. (As noted above, if the limit exists, then it must be equal to λ_κ.)
2. Set a good *upper* bound on λ_κ. (Traditionally, upper bounds are harder to obtain than lower bounds).

Other future research directions include a few subfamilies of TC polyominoes.

Definition 4 *(minimality) A TC polyomino P is* minimal *if no proper subset of cells of P is TC.*

Duplicating any row or column of a TC polyomino results in a TC polyomino. The opposite is also true: Discarding all but at least one of consecutive identical rows or columns of a TC polyomino results in a TC polyomino. This gives rise to the following definition.

Definition 5 *(primitivity) A TC polyomino is* primitive *if it does not contain any consecutive identical rows or columns.*

It is also worth considering TC polyominoes whose bounding boxes are "full."

Definition 6 *(saturation) A TC polyomino P is* saturated *if no empty cells in the bounding box of P can be filled and added to P, such that the result is still a TC polyomino.*

Here are some more questions to explore.

3. Are there members of the above subfamilies of unlimited size? (We found minimal, primitive, and saturated TC polyominoes of unlimited size.)

4. Is the intersection between the above subfamilies non-empty?

5. Do the sequences that enumerate the above subfamilies have growth constants? (For these subfamilies, we cannot apply concatenation arguments since the concatenation of pairs of minimal or saturated TC polyominoes always result in polyominoes which do not belong to these subfamilies, and the concatenation of pairs of primitive TC polyominoes might result in TC polyominoes which are not primitive.)

6. Design efficient algorithms for counting members of the above subfamilies. (At a first glance, it seems that extending Jensen's algorithm for any of the above subfamilies is unlikely since the properties defining the subfamilies are global.)

Further research directions involve more general settings of the problem.

7. Consider polyominoes in which each row and column contains at least $k > 2$ (say, 3) maximal sequences of occupied cells.

8. Explore similar problems in other planar lattices (*e.g.*, the triangular or hexagonal lattice).

9. Investigate similar problems for polycubes (face-connected sets of cells on cubical lattices) in higher dimensions. (Note the two possible *different* definitions of total concavity in a higher dimension d: A "weak" total concavity would require that every line parallel to one of the coordinate axes cross the polycube in either 0 or at least two maximal sequences of cells; A "strong" total concavity would require recursively (for $d > 2$) that the intersection of every $(d-1)$-dimenisonal hyperplane, perpendicular to one of the coordinate axes, be either empty or a $(d-1)$-dimenisonal TC polycube, where total concavity in two dimensions is as defined in this paper.)

References

[1] G. Barequet and G. Ben-Shachar, Counting polyominoes, revisited, *SIAM Symp. on Algorithm Engineering and Experiments*, Alexandria, VA, pp. 133–143, January 2024.

[2] G. Barequet, G. Ben-Shachar, and M.C. Osegueda, Concatenation arguments and their applications to polyominoes and polycubes, *Computational Geometry: Theory and Applications*, 98 (2021), 12 pp.

[3] G. Barequet and M. Moffie, On the complexity of Jensen's algorithm for counting fixed polyominoes, *J. of Discrete Algorithms*, 5 (2007), 348–355.

[4] G. Barequet, G. Rote, and M. Shalah, $\lambda > 4$: An improved lower bound on the growth constant of polyominoes, *Comm. of the ACM*, 59 (2016), 88–95.

[5] G. Barequet and M. Shalah, Improved upper bounds on the growth constants of polyominoes and polycubes, *Algorithmica*, 84 (2022), 3559–3586.

[6] G. Barequet, M. Shalah, and Y. Zheng, An improved lower bound on the growth constant of polyiamonds, *J. of Combinatorial Optimization*, 37 (2019), 424–438.

[7] G. Barequet, S.W. Solomon, and D.A. Klarner, Polyominoes, *Handbook of Discrete and Computational Geometry*, 3rd ed. (E. Goodman, J. O'Rourke, and C.D. Tóth, eds.), 359–380. Chapman and Hall/CRC Press, 2017.

[8] E.A. Bender, Convex n-ominoes, *Discrete Mathematics*, 8 (1974), 219–226.

[9] M. Bousquet-Mélou and J.M. Fédou, The generating function of convex polyominoes: The resolution of a q-differential system, *Discrete Mathematics*, 137 (1995), 53–75.

[10] S.R. Broadbent and J.M. Hammersley, Percolation processes: I. Crystals and mazes, *Proc. Cambridge Philosophical Society*, 53 (1957), 629–641.

[11] G. Castiglione and A. Restivo, Ordering and convex polyominoes, *Proc. 4th Int. Conf. on Machines, Computations, and Universality*, Saint Petersburg, Russia, September, 128–139, 2004 (Revised Selected Papers 4, Springer Berlin Heidelberg, 2005).

[12] S.W. Golomb, *Polyominoes*, Princeton University Press, Princeton, NJ, 1965 (2nd ed., 1994).

[13] W. Hochstättler, M. Loebl, and C. Moll, Generating convex polyominoes at random, *Discrete Mathematics* 153 (1996), 165–176.

[14] I. Jensen, Enumerations of lattice animals and trees, *J. of Statistical Physics*, 102 (2001), 865–881.

[15] I. Jensen, Counting polyominoes: A parallel implementation for cluster computing, *Proc. Int. Conf. on Computational Science*, part III, Melbourne, Australia and St. Petersburg, Russia, *Lecture Notes in Computer Science*, 2659, Springer, 203–212, June 2003.

[16] D.A. Klarner, Cell growth problems, *Canadian J. of Mathematics*, 19 (1967), 851–863.

[17] N. Madras, A pattern theorem for lattice clusters, *Annals of Combinatorics*, 3 (1999), 357–384.

[18] G. Pólya and G. Szegő, Aufgaben und Lehrsätze aus der Analysis Bd. Funktionentheorie. Nullstellen. Polynome. Determinanten. Zahlentheorie (Tasks and theorems from analysis: Vol. on Function theory, zeros, polynomials, determinants, number theory), vol. 2, Springer, 1925.

[19] C. Vanderzande, *Lattice models of polymers*, no. 11, Cambridge University Press, 1998.